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ON STABILITY CONDITIONS FOR THE NONISOTHERMIC COUETTE FLOW*

V. V. KOLESOV

The stability of flow of a viscous fluid contained between two rotating concentric cylinders heated to different temperatures is investigated. Necessary and sufficient conditions of stability (in the small) are indicated for such flow with respect to three-dimensional space-periodic perturbations, when the angular velocities of cylinders are equal. Stability and instability criteria are derived for the limit case of infinitely small clearance between the cylinders. It is shown that under certain conditions the loss of stability is related to monotonic perturbations.

1. Statement of the problem. Let a viscous homogeneous heat-conducting fluid be contained between two infinite solid concentric cylinders. We denote the radii, angular velocities, and temperatures of the inter an outer cylinders, respectively, by R_1, Ω_1, Θ_1 and R_2, Ω_2, Θ_2. We assume that external mass forces are absent and that the rate of flow through the cross section of the intercylinder space is zero. As the scales of length, velocity, temperature, and density we take $R_1, \Omega_1 R_1, \Theta_1$, and the fluid density at temperature Θ_1.

In the cylindrical system of coordinates r, φ, z (the z-axis coincides with the axis of cylinders) the Navier—Stokes equations of continuity, heat conduction, and of state are of the form

$$\frac{d\mathbf{V}'}{dt} = -\frac{1}{\rho'}\nabla\Pi' + \frac{1}{\lambda}(\nabla\operatorname{div}\mathbf{V}' - \operatorname{rot}\operatorname{rot}\mathbf{V}'), \quad \frac{\partial\rho'}{\partial t} + \operatorname{div}(\rho'\mathbf{V}') = 0 \tag{1.1}$$

$$\frac{\partial T'}{\partial t} = \frac{1}{\lambda P}\Delta T' - (\mathbf{V}', \nabla)T', \quad \rho' = 1 - \beta\Theta_1(T' - 1)$$

$$\int_0^{2\pi}\int_1^{R_2/R_1} v_z' r\,dr\,d\varphi = 0; \quad \mathbf{V}' = \{v_r', v_\varphi', v_z'\} \quad \lambda = \frac{\Omega_1 R_1^2}{\nu}, \quad P = \frac{\nu}{\chi}, \quad \Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2},$$

$$\nabla = \left\{\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial\varphi}, \frac{\partial}{\partial z}\right\}, \quad \frac{d\mathbf{V}'}{dt} = \frac{\partial\mathbf{V}'}{\partial t} + (\mathbf{V}', \nabla)\mathbf{V}' + \left\{-\frac{1}{r}(v_\varphi')^2, \frac{1}{r}v_r'v_\varphi', 0\right\}$$

The boundary conditions $\mathbf{V}' = \{0, 1, 0\},\ T' = 1,\ r = 1, \quad \mathbf{V}' = \{0, \Omega R, 0\},\ T' = \Theta,\ r = R$ (1.2)

$$(R = R_2 / R_1,\ \Omega = \Omega_2 / \Omega_1,\ \Theta = \Theta_2 / \Theta_1)$$

must be satisfied at the cylinder surfaces.

Problem (1.1), (1.2) admits an exact solution (the Couette circular laminar nonisothermic flow with logarithmic temperature distribution)

$$\mathbf{V}_0 = \{0, ar + b/r, 0\},\ T_0 = c\ln r + 1, \quad \Pi_0 = \int_1^r (a + b/x^2)^2(1 - \beta c\Theta_1\ln x)\,x\,dx + \text{const} \tag{1.3}$$

$$a = (\Omega R^2 - 1)/(R^2 - 1),\ b = 1 - a,\ c = (\Theta - 1)/\ln R$$

We impose on the flow (1.3) the infinitely small perturbations

$$\mathbf{V}' = \mathbf{V}_0 + \mathbf{V},\ T' = T_0 + cPT,\ \Pi' = \Pi_0 + \Pi/\lambda \tag{1.4}$$

and substituting (1.4) into (1.1) and (1.2), and linearizing the obtained problem in the neighborhood of flow (1.3), we obtain the stability problem

$$\frac{\partial v_r}{\partial t} + \omega_1\frac{\partial v_r}{\partial\varphi} - \frac{1}{\lambda}\left(\Delta v_r - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\varphi}{\partial\varphi}\right) + \frac{1}{\lambda}\frac{\partial\Pi}{\partial r} = 2\omega_1 v_\varphi - \mathrm{Ra}\,\omega_2 T \tag{1.5}$$

$$\frac{\partial v_\varphi}{\partial t} + \omega_1\frac{\partial v_\varphi}{\partial\varphi} - \frac{1}{\lambda}\left(\Delta v_\varphi - \frac{v_\varphi}{r^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial\varphi}\right) + \frac{1}{\lambda r}\frac{\partial\Pi}{\partial\varphi} = g_1 v_r, \quad \frac{\partial v_z}{\partial t} + \omega_1\frac{\partial v_z}{\partial\varphi} - \frac{1}{\lambda}\Delta v_z + \frac{1}{\lambda}\frac{\partial\Pi}{\partial z} = 0$$

$$\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\varphi}{\partial\varphi} + \frac{\partial v_z}{\partial z} = 0, \quad \frac{\partial T}{\partial t} + \omega_1\frac{\partial T}{\partial\varphi} - \frac{1}{\lambda P}\Delta T = -\frac{g_2}{P}v_r, \quad \int_0^{2\pi}\int_1^R v_z r\,dr\,d\varphi = 0$$

$$v_r = v_\varphi = v_z = T = 0 \quad (r = 1, r = R), \ (\mathrm{Ra} = \beta c\Theta_1 P,\ \omega_1 = a + b/r^2,\ \omega_2 = \omega_1^2 r,\ g_1 = -2a,\ g_2 = 1/r)$$

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*Prikl.Matem.Mekhan.,44,No.3,441-447,1980 which is represented here in the Boussinesq approximation $\beta\Theta_1 \leqslant 1$ /1/.

Note that the validity of linearization of the stability problem (1.5), i.e. the admissibility of assessing the stability (in the small) of flow (1.3) on the basis of the analysis of the linearized stability problem (1.5) was justified in /2,3/.

Henceforth we assume that perturbations of $\mathbf{V}$, T, and Π are periodic in the azimuthal and axial directions of specified periods $2\pi / m$ and $2\pi / \alpha$, respectively, with $m = 0, 1, 2, \ldots$; $\alpha \geqslant 0$; $m^2 + \alpha^2 \neq 0$.

2. Stability relative to three-dimensional perturbations.

The following two theorems establish the necessary and sufficient stability conditions for the flow (1.3) relative to three-dimensional perturbations of arbitrary periodicity, when the cylinders rotate at the same angular velocities. The proof is based on the methods by which was obtained the simple proof of the Singe criterion $(\Omega R^2 - 1 \geqslant 0)$ /4/ of the Couette isothermic flow stability relative to rotationally symmetric perturbations (independent of φ).

Theorem 2.1. Let us consider the case when the two cylinders rotate at the same angular velocities $(\Omega = 1)$ and the gap between the cylinders is not too large $(\ln R \leqslant \pi)$. For the flow (1.3) to be stable relative to three-dimensional infinitely small periodic perturbations at any Reynolds numbers $\lambda > 0$ it is sufficient that the outer cylinder temperature does not exceed that of the inner cylinder $(\mathrm{Ra} \leqslant 0)$.

Proof. Let $\Omega = 1$. Multiplying the first equation of system (1.5) by v_r, the second by v_φ, the third by v_z, and the fifth by $-P\,\mathrm{Ra}\, r^2 T$, adding the obtained equalities and integrating over the region $D: \{1 \leqslant r \leqslant R,\ |\varphi| \leqslant \pi / m,\ |z| \leqslant \pi / \alpha\}$ with weight r, we obtain

$$\frac{1}{2}\frac{\partial}{\partial t}\int\limits_D (v_r^2 + v_\varphi^2 + v_z^2 - P\,\mathrm{Ra}\, r^2 T^2)\, r\, dr\, d\varphi\, dz = -(J_1 + J_2)/\lambda + \mathrm{Ra}\,(J_3 + J_4)/\lambda \tag{2.1}$$

$$J_1 = \int\limits_D \left[\left(\frac{\partial v_r}{\partial \varphi} - v_\varphi\right)^2 + \left(\frac{\partial v_\varphi}{\partial \varphi} + v_r\right)^2 + \left(\frac{\partial v_z}{\partial \varphi}\right)^2\right]\frac{1}{r}\, dr\, d\varphi\, dz$$

$$J_2 = \int\limits_D \left[\left(\frac{\partial v_r}{\partial r}\right)^2 + \left(\frac{\partial v_r}{\partial z}\right)^2 + \left(\frac{\partial v_\varphi}{\partial r}\right)^2 + \left(\frac{\partial v_\varphi}{\partial z}\right)^2 + \left(\frac{\partial v_z}{\partial r}\right)^2 + \left(\frac{\partial v_z}{\partial z}\right)^2\right] r\, dr\, d\varphi\, dz$$

$$J_3 = \int\limits_D \left[\left(\frac{\partial T}{\partial \varphi}\right)^2 + r^2\left(\frac{\partial T}{\partial z}\right)^2\right] r\, dr\, d\varphi\, dz, \qquad J_4 = \int\limits_D \left[\left(\frac{\partial T}{\partial r}\right)^2 + \frac{1}{r}\frac{\partial}{\partial r}(T^2)\right] r^3 dr\, d\varphi\, dz$$

Let us consider the functional

$$J(\tau) = \int\limits_1^R \left[\left(\frac{d\tau}{dr}\right)^2 + \frac{1}{r}\frac{d}{dr}(\tau^2)\right] r^3 dr$$

on the set of smooth real functions $\tau(r)$ which vanish for $r = 1$ and $r = R$. Using the variational principle for the Sturm—Liouville problem

$$-\frac{1}{r}\frac{d}{dr}\left(r^3\frac{d\tau}{dr}\right) = \varkappa\tau, \quad \tau(1) = \tau(R) = 0$$

we find that

$$\min_\tau \left[\int\limits_1^R \left(\frac{d\tau}{dr}\right)^2 r^3 dr \left(\int\limits_1^R \tau^2 r dr\right)^{-1}\right] = \frac{\pi^2}{\ln^2 R} + 1 \tag{2.2}$$

where the minimum for the function

$$\tau = \frac{1}{r}\sin\frac{\pi \ln r}{\ln R}$$

Taking into account that according to the condition $\ln R \leqslant \pi$, we have from (2.2) that the functional $J(\tau)$ is nonnegative. But then the right-hand side of equality (2.1) is negative and the flow (1.3) is stable for any $\lambda > 0$ and $\mathrm{Ra} \leqslant 0$. The theorem is proved.

Theorem 2.2. Let $\Omega = 1$. If $\mathrm{Ra} > 0$, then the flow (1.3) looses its stability for fairly large λ.

Proof. It suffices to show that for some λ the spectrum of stability of flow (1.3) contains an eigenvalue σ with a positive real part. We set in (1.5)

$$\frac{rv_r}{u(r)} = \frac{v_\varphi}{v(r)} = \frac{v_z}{w(r)} = \frac{T}{\tau(r)} = \frac{\Pi}{q(r)} = e^{\sigma t + im\varphi} \tag{2.3}$$

The substitution of (2.3) into (1.5) yields when $\Omega = 1$, after the separation of variables, the spectral problem

$$[L_m - \lambda(\sigma + im)]\, L_m u = -m^2\lambda\, \mathrm{Ra}\, \tau, \qquad [L_m - \lambda P(\sigma + im)]\, \tau = \lambda u / r^2, \quad L_m = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2} \tag{2.4}$$

$$\frac{du}{dr} = u = \tau = 0 \quad (r = 1, r = R)$$

We set $\sigma = \sigma^{\mathrm{Re}} - i$, where σ^{Re} is any positive number, and shall show that this eigenvalue σ for $m = 1$ and fairly large λ belongs to the spectrum of problem (2.4).

When $m = 1$ and $\sigma = \sigma^{\mathrm{Re}} - i$, the operators in the left-hand side of system (2.4) can be represented as

$$-r(L_1 - \lambda P\sigma^{\mathrm{Re}}) = -\varkappa_0 \frac{d}{dr}\varkappa_1 \frac{d}{dr}\varkappa_2, \qquad r(L_1 - \lambda\sigma^{\mathrm{Re}})\, L_1 = \varkappa_3 \frac{d}{dr}\varkappa_4 \frac{d}{dr}\varkappa_5 \frac{d}{dr}\varkappa_6 \frac{d}{dr}\varkappa_7 \tag{2.5}$$

$$\varkappa_0 = \varkappa_2 = 1/I_1(r\sqrt{\lambda P\sigma^{\mathrm{Re}}}), \quad \varkappa_1 = rI_1^2(r\sqrt{\lambda P\sigma^{\mathrm{Re}}})$$
$$\varkappa_3 = \varkappa_5 = 1/I_1(r\sqrt{\lambda\sigma^{\mathrm{Re}}}), \quad \varkappa_4 = rI_1^2(r\sqrt{\lambda\sigma^{\mathrm{Re}}}), \quad \varkappa_6 = 1/r, \quad \varkappa_7 = r$$

where I_1 are modified Bessel functions of the first kind.

With $m = 1$ and $\sigma = \sigma^{\mathrm{Re}} - i$ problem (2.4) is equivalent to the integral equation

$$u = \mu Bu, \ \mu = \lambda^2\mathrm{Ra}, \qquad (Bu)(r) = \int_1^R G_2(r, \rho) \int_1^R G_1(\rho, s)\frac{1}{s}u(s)\, ds\, \rho\, d\rho \tag{2.6}$$

where $G_1(r, \rho)$ and $G_2(r, \rho)$ are Green's functions of differential operators defined by the first and second of equalities (2.5) with conditions $\tau = 0$ and $du/dr = u = 0$ $(r = 1, r = R)$, respectively.

On the basis of results in /5,6/, using (2.5) and (2.6) we conclude that B is an oscillating operator. But then there exists a sequence of values of $\mu: \mu_1 < \mu_2 < \ldots;\ \mu_n \to \infty$ such that Eq. (2.6) has a nontrivial solution and, consequently, the flow (1.3) looses stability for fairly large λ. The theorem is proved.

3. Stability with respect to two-dimensional perturbations.

Below, we consider the stability of flow (1.3) with respect to plane and rotational symmetric perturbations (independent of z and φ, respectively) in the case of an infinitely small clearance between the cylinders. The sufficient conditions of stability of flow (1.3) with respect to rotation symmetric perturbations for arbitrary clearance between cylinders appear in /7/.

Let us assume that the clearance between cylinders is infinitely small $(R \to 1)$, the cylinders rotate at the same angular velocities $(\Omega = 1)$, and the perturbations are plane. Setting in (1.5)

$$\frac{v_r}{u(x)} = \frac{v_\varphi}{v(x)} = \frac{v_z}{w(x)} = \frac{T}{\tau(x)} = \frac{\Pi}{q(x)} = e^{\sigma t + imy}, \qquad x = \frac{r-1}{R-1}, \quad y = \frac{\varphi - t}{R-1}, \quad \lambda_1 = (R-1)^2\lambda$$

and neglecting terms of order $R - 1$, after separation of variables, we obtain the spectral problem

$$(\Lambda_m - \lambda_1\sigma)\Lambda_m u = -m^2\lambda_1\, \mathrm{Ra}\, \tau, \qquad (\Lambda_m - \lambda_1\sigma P)\, \tau = \lambda_1 u, \quad \Lambda_m = d^2/dx^2 - m^2 \tag{3.1}$$

$$du/dx = u = \tau = 0 \quad (x = 0, x = 1)$$

which except for the notation, coincides with the problem of convection onset in a plane horizontal fluid layer between solid boundaries. Using this analogy it is possible to verify that the eigenvalues σ of problem (3.1) which lie in the right half-plane $(\sigma^{\mathrm{Re}} \geqslant 0)$ are real for any $\lambda_1 > 0$.

It follows from this that in the coordinate system which rotates at the same angular velocity as the cylinders, the plane secondary flow induced by the loss of stability the flow (1.3) is stable. In the stationary coordinate system the plane secondary flow is self-oscillating of the type of azimuthal waves whose phase velocity is equal unity.

Let us consider the case of rotating symmetric perturbations. We assume that the clearance between cylinders is infinitely small and that the cylinders rotate at infinitely close angular velocities, with $M = (\Omega - 1)/(R - 1)$ bounded as $R \to 1$ and $\Omega \to 1$.

Setting

$$v_r = u(x)\, e^{\sigma t}\cos\alpha z, \ v_\varphi = v(x)\, e^{\sigma t}\cos\alpha z, \qquad v_z = w(x)\, e^{\sigma t}\sin\alpha z, \ T = \tau(x)\, e^{\sigma t}\cos\alpha z$$

$$\Pi = q(x)\, e^{\sigma t}\cos\alpha z, \quad x = (r-1)/(R-1), \quad \lambda_1 = (R-1)^2\lambda$$

and neglecting terms of order $R - 1$ and $\Omega - 1$, after separating variables we obtain the spectral problem

$$(\Lambda_\alpha - \lambda_1\sigma)\,\Lambda_\alpha u = \alpha^2\lambda_1(2v - \mathrm{Ra}\,\tau), \qquad (\Lambda_\alpha - \lambda_1\sigma)\, v = \lambda_1(M + 2)\, u, \ (\Lambda_\alpha - \lambda_1\sigma P)\, \tau = \lambda_1 u \tag{3.2}$$

$$du/dx = u = v = \tau = 0 \quad (x = 0, x = 1)$$

Theorem 3.1. If the inequalities

$$\mathrm{Ra} \leqslant 0, \ M + 2 > 0 \tag{3.3}$$

are satisfied, then for any Reynolds number $\lambda_1 > 0$ all eigenvalues σ of problem (3.2) lie within the left half-plane ($\sigma^{\mathrm{Re}} < 0$). If, however, the inequalities

$$\mathrm{Ra} \geqslant 0, \ M + 2 < 0 \tag{3.4}$$

are satisfied, then at fairly large λ_1 the spectrum of problem (3.2) contains at least one eigenvalue σ inside the right half-plane ($\sigma^{\mathrm{Re}} > 0$).

Proof. Let inequalities (3.3) be satisfied. Multiplying the first equation of system (3.2) by $\bar{u}$, the second by $\bar{v}$, and the third by $\bar{\tau}$, integrating with respect to x from 0 to 1 and separating in the obtained equalities the real and imaginary parts, we obtain

$$J_1 + \frac{2\alpha^2}{M+2} J_2 - \alpha^2 \mathrm{Ra}\, J_3 + \lambda_1 \sigma^{\mathrm{Re}} \Big(J_4 + \frac{2\alpha^2}{M+2} J_5 - \alpha^2 \mathrm{Ra}\, P J_6\Big) = 0\,, \quad \lambda_1 \sigma^{\mathrm{Im}} \Big(J_4 - \frac{2\alpha^2}{M+2} J_5 + \alpha^2 \mathrm{Ra}\, P J_6\Big) = 0 \tag{3.5}$$

where J_k ($k = 1, 2, \ldots, 6$) are nonnegative functionals. On the basis of conditions (3.3) we obtain from (3.5) $\sigma^{\mathrm{Re}} < 0$.

Let now inequalities (3.4) be satisfied. By inverting the operators in the left-hand sides of system (3.2) using Green's functions, we reduce problem (3.2) for any $\sigma > 0$ to the single integral equation

$$u = \alpha^2 \lambda_1^2 B u, \ B = B_1 + B_2, \quad (B_1 u)(x) = \mathrm{Ra} \int_0^1 G_1(x, y) \int_0^1 G_3(y, s)\, u(s)\, ds\, dy \tag{3.6}$$

$$(B_2 u)(x) = -2(M+2) \int_0^1 G_1(x, y) \int_0^1 G_2(y, s)\, u(s)\, ds\, dy$$

where $G_1(x, y), G_2(x, y)$ and $G_3(x, y)$ are Green's functions of the differential operators $(\Lambda_\alpha - \lambda_1 \sigma)\Lambda_\alpha$, $-(\Lambda_\alpha - \lambda_1 \sigma)$ and $-(\Lambda_\alpha - \lambda_1 \sigma P)$ with boundary conditions $du / dx = u = 0$, $v = 0$ and $\tau = 0$ ($x = 0$, $x = 1$), respectively.

Using the results of /5,6/ we find that operators B_1 and B_2 are oscillatory. Then applying the data in /9/ and taking into account inequalities (3.4), we obtain the following statement. For any $\sigma > 0$ operator B is u_0-positive in the cone of nonnegative functions and, consequently, there exists a λ_1 such that the integral equation (3.6) has a nontrivial solution. The latter means that the spectrum of problem (3.2) contains at some λ_1 a positive eigenvalue σ. The theorem is proved.

Remarks 1°. A slight change in the proof enables us to establish that when $M + 2 = 0$ the whole spectrum of problem (3.2) lies inside the left half-plane at any $\mathrm{Ra} \leqslant 0, \lambda_1 > 0$, while for $\mathrm{Ra} > 0$ and fairly large λ_1 part of the spectrum is in the right half-plane, and the whole /part of the / spectrum lying in that half-plane is real.

2°. It follows from Theorem 3.1 and Remark 1° that at the limit $R \to 1, \Omega \to 1$ the fulfillment of inequality (3.4) is sufficient for stability of the flow (1.3) with respect to rotating symmetric perturbations of arbitrary periodicity and any $\lambda_1 > 0$, in order that inequalities (3.3) are fulfilled, while for the instability of flow (1.3) at some λ_1 the fulfillment of condition (3.4) is sufficient. If $M + 2 = 0$, then for stability of the flow (1.3) at any $\lambda_1 > 0$ it is necessary and sufficient to fulfil the conditions $\mathrm{Ra} \leqslant 0$, and the secondary rotating symmetric flow induced by the loss of stability of flow (1.3) is steady.

Theorem 3.2. Let us assume that the Prandtl number $P = 1$. If all eigenvalues σ of problem(3.2) are to be contained within the left half-plane ($\sigma^{\mathrm{Re}} < 0$) at any $\lambda_1 > 0$, it is necessary and sufficient that the inequality $\mathrm{Ra} \leqslant 2(M + 2)$ is satisfied. The eigenvalues σ that lie in the right half-plane ($\sigma^{\mathrm{Re}} \geqslant 0$) are real.

Proof of this theorem is similar to that of Theorem 3.1. Note that when $P = 1$, the relation $v \equiv (M + 2)\tau$ is satisfied, which results in considerable simplifications.

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